CHAPTER 1

A Critique of Absolutism in the Philosophy of Mathematics

Historically, mathematics has long been viewed as the paradigm of infallibly secure knowledge. Euclid and his colleagues first constructed a magnificent logical structure around 2,300 years ago in the *Elements*, which at least until the end of the nineteenth century was taken as the paradigm for establishing incorrigible truth. Descartes ([1637] 1955) modeled his epistemology directly on the method and style of geometry. Hobbes claimed that "geometry . . . is the only science . . . bestow[ed] on [hu]mankind" (Hobbes [1651] 1962, 77). Newton in his *Principia* and Spinoza in his *Ethics* used the form of the *Elements* to strengthen their claims of systematically expounding the truth.[1] This logical form reached its ultimate expression in *Principia Mathematica*, in which Whitehead and Russell (1910–13) reapplied it to mathematics, while paying homage to Newton with their title. As part of the logicist program, *Principia Mathematica* was intended to provide a rigorous and certain foundation for all of mathematical knowledge. Thus mathematics has long been taken as the source of the most infallible knowledge known to humankind, and much of this is due to the logical structure of its presentation and justification.

With this background, a philosophical inquiry into mathematics raises questions including: What is the basis for mathematical knowledge? What is the nature of mathematical truth? What characterizes the truths of mathematics? What is the justification for their assertion? Why are the truths of mathematics necessary truths? How absolute is this necessity?

The Nature of Knowledge

The question, What is knowledge? lies at the heart of philosophy, and mathematical knowledge plays a special part. The standard philosophical answer, which goes back to Plato, is that knowledge is justified true belief. To put it differently, propositional knowledge consists of propositions which are accepted (i.e., believed), provided there are adequate grounds fully available to the believer for asserting them (Sheffler 1965; Chisholm 1966; Woozley 1949). This way of putting it avoids presupposing the truth of what is known, although traditional accounts require it, by referring instead to adequate grounds, which also include the justificatory element. The phrase "fully available" circumvents the difficulty caused when the adequate grounds exist but are not in the cognizance of the believer.[2]

Knowledge is classified on the basis of the grounds for its assertion. *A priori* knowledge consists of propositions which are asserted on the basis of reason alone, without recourse to observations of the world. Here reason consists of the use of deductive logic and the meanings of terms, typically to be found in definitions. In contrast, empirical or *a posteriori* knowledge consists of propositions asserted on the basis of experience, that is, based on observations of the world (Woozley 1949). This basis refers strictly to the empirical *justificatory* basis of a posteriori knowledge, not its genesis. Indeed, such knowledge may be initially generated by pure thought, whilst a priori knowledge, such as that of mathematics, may be first generated by induction from empirical observation. Such origins are immaterial; only the grounds for asserting the knowledge matter. This distinction is first to be found in Kant ([1781] 1961), but also occurs implicitly in earlier work, such as in Leibniz ("truths of reason" versus "truths of fact") and Hume ("matters of fact" versus "matters of reason"), Vico ("verum" or a priori truth versus "certum" or the empirical), as well as being anticipated by Plato.

Kant not only distinguishes a priori and a posteriori knowledge, on the basis of the means of verification used to justify them, but also distinguishes between *analytic* and *synthetic* propositions. A proposition is *analytic* if it follows from the law of contradiction, that is, if its denial is logically inconsistent.[3] Kant argued that mathematical knowledge is synthetic *a priori*, since it is based on reason, not empirical facts, but does not follow from the law of contradiction alone. The standard view in epistemology (see Feigl and Sellars 1949, for example) is that Kant was wrong and mathematics is analytic, and that the analytic can be identified with the *a priori* and the synthetic with the *a posteriori*. According to this view, mathematical theorems add nothing to knowledge which is not implicitly contained in the premises *logically*, although *psychologically* the theorems may be novel.

The debate is not straightforward, for a number of reasons. First of all, Kant believed in a universal logic, whereas now we recognize alternative systems in logic (Haack 1974, 1978). He also believed that mathematical theories such as Euclidean geometry and arithmetic are the necessary logical outcomes of reason. (Non-Euclidean geometry and nonstandard arithmetics were simply not possible in his system.) He concluded that although the truths of mathematics are necessary, they do not follow from the law of contradiction, but from the forms that human understanding takes, by its very nature.

A number of modern philosophers have agreed with Kant, at least so far as to dissent from the received view that identifies the analytic with the *a priori* and the synthetic with the *a posteriori*. Hintikka (1973) argues that some mathematical proofs require the addition of auxiliary elements or concepts, and hence add something unforeseen and logically novel to the mathematical knowledge. Since such proofs do not rest on the law of contradiction alone, he argues that they are synthetic, in both senses, as well as *a priori*. Brouwer and Wittgenstein (as I shall show below and in chap. 3, respectively) similarly accept that some mathematical knowledge is both synthetic and *a priori*. Finally, some others, such as Quine (1953b, 1970) and White argue that "the analytic and the synthetic [is] an untenable dualism" (White 1950). Their view is that the boundary between the two classes cannot be fixed determinately. Quine (1960) goes on to elaborate his view that mathematical and empirical scientific knowledge cannot be neatly partitioned into the analytic and synthetic. He argues that the whole of language is a "vast verbal structure," and it is not possible to separate out those parts which have empirical import from those that do not; "this structure of interconnected sentences is a single connected fabric including all sciences, and . . . logical truths" (Quine 1960, 12).

These subtleties and dissenting views notwithstanding, according to the received view mathematical knowledge is classified as *a priori* knowledge, since it consists of propositions asserted on the basis of reason alone. Reason includes deductive logic and definitions which are used, in conjunction with an assumed set of mathematical axioms or postulates, as a basis from which to infer mathematical knowledge. Thus the foundation of mathematical knowledge, that is, the grounds for asserting the truth of mathematical propositions, consists of deductive proof, together with the assumed truth of any premises employed. Apart from the assumed truth of the premises, there is another fundamental way in which mathematical proof depends on truth. The essential underpinning feature of a correct or valid deductive proof is the transmission of truth, that is, truth value is preserved.

Truth in Mathematics

It is often the case in mathematics that the definition of truth is assumed to be clear-cut, unambiguous, and unproblematic. While this is often justifiable as a simplifying assumption, the fact is that it is incorrect and that the meaning of the concept of truth in mathematics has changed significantly over time. I wish to distinguish among three truth-related concepts used in mathematics.

The traditional view of mathematical truth. First of all, there is the traditional view that a mathematical truth is a general statement which not only correctly describes all its instances in the world (as would a true empirical generalization) but is *necessarily* true of its instances. Implicitly underpinning this view is the assumption that mathematical theories have an intended interpretation, often an idealization of some aspect of the world. The key feature of this view is the association of an intended interpretation with a theory. Thus number theory refers to the domain of natural numbers, geometry refers to ideal objects in space, calculus largely refers to functions of the real line, and so on. To be true in this first sense (I will denote it by "truth$_1$") is to be true in the intended interpretation. The mode of expression I have used depends of course upon a modern way of thinking, for it requires prizing open mathematical signs to separate the signifiers (formal mathematical symbols) from the signified (the intended meanings). Truth$_1$ treats mathematical signs as integral; only one interpretation is built in.

Truth$_1$ is analogous to naive realism, a view of truths as statements which accurately describe a state of affairs in some fixed realm of discourse. According to this view, the terms involved in expressing the truth name objects in the intended universe of discourse, and the true statement as a whole describes the relationship that holds between these denotations. In essence, this is the naive correspondence theory of truth.

Such a view of mathematical truth was widespread, dominant even, until the middle and end of the nineteenth century. For example De Morgan commenting on Peacock's new generalized formal algebra described it as made up of "symbols bewitched . . . running about the world in search of meaning" (1835, 311). What he objected to was the severance of algebraic symbols from their generalized arithmetical meanings (Richards 1987). Without such fixed and determinate meanings, mathematical propositions could not express their intended meanings, let alone truths. Similarly, Frege had a sophisticated and philosophically well elaborated view that the theorems of arithmetic are true in its intended interpretation, the domain of natural number. Again, this is the notion of truth$_1$.

Mathematical truth as satisfiability. Secondly, there is the modern view of the truth of a mathematical statement relative to a background mathematical theory: the statement is satisfied by some interpretation or model of the theory. I shall term this second conception "truth$_2$." According to this (and the following) view, mathematical theories are open to multiple interpretations, that is, possible worlds. Truth in this sense consists merely in being true (i.e., satisfied, following Tarski 1936) in one of these possible worlds; that is, in having a model. Thus truth$_2$ is represented by Tarski's explication of truth, which forms the basis of model theory. A proposition is true$_2$ relative to a given mathematical theory if there is some interpretation of the theory which satisfies the proposition, irrespective of the other properties of the interpretation, such as resemblance to some original intended interpretation. (This interpretation must include an assignment of objects and relations of appropriate type to the extralogical symbols, as well as an assignment of values from the universe of discourse to the variable letters of the proposition.)

Truth$_2$ probably originates with Hilbert's work on geometry. Hilbert detached geometrical notions such as 'point, line, and plane' from their original physical (or ideal) interpretations, and argued instead that they could be interpreted as 'table, chair, and beer-mug', provided that what resulted was a model of the axioms of geometry. It has been suggested that Tarski's theory of truth originates in algebra, by analogy with a set of roots satisfying an equation. Likewise, the assignment of values to the components of a proposition satisfies it when it makes it true.

Truth$_2$ is anticipated by Leibniz's notion of 'true in a possible world', which he contrasted with 'true in all possible worlds' (Barcan Marcus 1967).

Logical truth or validity in mathematics. Thirdly, there is the modern view of the logical truth or validity of a mathematical statement relative to a background theory: the statement is satisfied by *all* interpretations or models of the theory. Thus the statement is true in all of these representations of possible worlds. I shall denote this conception of truth by 'truth$_3$'. Evidently truth$_3$ more or less corresponds to Leibniz's notion of 'true in all possible worlds'. This is also one of the notions explicated by Tarski's theory of mathematical truth as 'logical validity'.

Truth$_3$ can be established by logical deduction from the background theory if the theory is represented by a first-order axiom set, as Gödel's (1930) completeness theorem establishes. For a given theory, Truths$_3$ (the set of propositions which are true in the sense of truth$_3$) is a subset (usually a proper subset) of Truths$_2$. Incompleteness arises, as Gödel ([1931] 1967) proved, in most mathematical theories as there are true$_1$ sentences (i.e., satisfied in the intended model) which are not true$_3$ (i.e., true in *all* models).

Thus not only does the concept of truth have multiple meanings, but crucial mathematical issues hinge upon this ambiguity. The modern mathematical views of truth (truth$_2$ and truth$_3$) differ in meaning and properties from the traditional mathematical view of truth$_1$ and the everyday naive notion which resembles it. Historically, the transition from truth$_1$ to the modern notions was highly problematic, as Richards (1980, 1989) shows in her studies. Even the correspondence between such mathematically (and philosophically) great thinkers as Frege (1980) and Hilbert shows disagreements and sometimes a lack of understanding that may be attributed to Frege's use of truth$_1$ and Hilbert's use of truth$_2$.

A consequence of this is that the traditional problem of establishing the indubitable foundations of mathematical truth has changed in meaning, as the definition of truth employed has changed. The relationship between the three notions explicated above is as follows (assuming a given background mathematical theory). Given any proposition P, if P is $true_3$, then P is also $true_1$; and if P is $true_1$, it is also $true_2$. Thus to claim that a statement is $true_2$ is much weaker than $truth_1$ or $truth_3$.

Although there are these complexities in the mathematical concept of truth, one way to vouchsafe it has remained at the center of mathematics for more than two millennia, that is, mathematical proof. This ties in with the discussion of truth, because as mentioned above provability (relative to a given set of axioms) is equivalent to $truth_3$ (Gödel 1930). Similarly, it follows from a contrapositive argument that consistency (relative to a given set of axioms) is equivalent to $truth_2$.

Proof in Mathematics

Since proof constitutes the means of justifying knowledge in mathematics, it is important to analyze how it does this. The proof of a mathematical proposition is a finite sequence of statements ending in the given proposition, which sequence ideally satisfies the following property.[4] Each statement is an axiom drawn from a previously stipulated set of axioms, or is derived by a rule of inference from one or more statements occurring earlier in the sequence. The term *set of axioms* should be understood broadly, to include whatever statements are admitted into a proof without demonstration, including axioms, postulates, and definitions.

This account describes a "primitive" and ideal proof, one in which all of the assumptions are primitive, that is, basic assumptions, and all of the inferences are justified by specified rules. In a "derived" proof some of these assumptions are themselves the results of earlier proofs. A derived proof can, in principle, be turned into a primitive proof simply by incorporating within it the proofs of all nonprimitive assumptions, and iterating this procedure until no nonprimitive assumptions remain. Thus there is no loss of generality in considering only primitive proofs.[5] However the assumption that all proofs can be rendered as ideal proofs, that is, as based on logical or mathematical rules of inference, is not so easily discharged.[6]

The idea underpinning the notion of proof is that of truth transmission. If the axioms adopted are taken to be true, and if the rules of inference infallibly transmit truth (i.e., true premises necessitate a true conclusion), then the theorem proved must also be true. For there is an unbroken and undiminished flow of truth from the axioms transmitted through the proof to the conclusion. With this in mind, the modern definitions of the logical connectives are understood in terms of truth tables. Thus, an implication statement $P \rightarrow Q$ is true if, and only if, it cannot be the case that P is true and Q is false. Thus to safeguard the transmission of truth in proof, the shared content or causal link between antecedent and consequent sometimes found in the everyday language usage of implication statements is sacrificed.

As a simple example of a mathematical proof I will analyze a proof of the statement $1 + 1 = 2$ in the axiomatic system of Peano arithmetic. This proof requires as assumptions a number of definitions and axioms, as well as logical rules of inference. These assumptions are the definitions of 1 and 2 as successors of 0 and 1, respectively, axioms specifying the properties of addition recursively, and logical rules stating that (1) two equal terms have the same properties and (2) a general property of numbers applies to any particular number. Based on these assumptions, $1 + 1 = 2$ can be proved in ten steps.[7] Each equation in the proof either is a specified assumption or is derived from earlier parts of the proof by applying rules of inference. Since the assumptions are assumed to be true, and the rules transmit truth, every equation in the sequence is equally true, including $1 + 1 = 2$. The proof establishes $1 + 1 = 2$ as an item of mathematical knowledge or truth, according to the previous analysis, for the deductive proof provides a legitimate warrant for asserting the statement.[8] Furthermore it is a priori knowledge, since it is asserted on the basis of reason alone.

However, what has not yet been made clear are the grounds for the assumptions made in the proof. These are of two types: mathematical and logical assumptions. The mathematical assumptions used are the definitions and the axioms. The logical assumptions are the rules of inference used, which are part of the overall proof theory, as well as the underlying syntax of the formal language. Although not specified here, this syntax is not negligible. It includes the categories of symbols, and the inductively defined rules of combination (e.g., for terms and sentences) and of transformation (e.g., substitution of individual terms in formulas).

I consider first the mathematical assumptions. Explicit mathematical definitions are unproblematic, since they are eliminable in principle. Thus every occurrence of the defined terms 2 and 1 can be replaced by what is abbreviated (the successors of 1 and 0, respectively), until these terms are completely eliminated. The result is an abbreviated proof of "the successor of zero plus the successor of zero = the successor of the successor of zero," which represents $1 + 1 = 2$ in other words. Although explicit definitions are eliminable in principle—that is, they do not entail any additional logical assumptions—they play an important (probably essential) role in human knowing. However, in the present context I am concerned to minimize assumptions, to reveal the irreducible assumptions on which mathematical knowledge and its justification rests.

If the definitions had not been explicit, such as in Peano's original inductive definition of addition (Heijenoort 1967), which are specified in the example as basic axioms (Ernest 1991, 5), then the definitions would not be eliminable in principle. This case is analogous to that of an axiom. In other words, a basic assumption would have been made and would have to be acknowledged as such.

I have now disposed of all the categories of assumption that are eliminable. The axioms in the proof are not eliminable. They must either be assumed as self-evident axiomatic (or otherwise warranted) truths or simply retain the status of unjustified, tentative assumptions, adopted to permit the development of the mathematical theory under consideration. The logical assumptions, that is, the rules of inference (part of the overall proof theory) and the logical syntax, are assumed as part of the underlying logic and are part of the mechanism needed for the application of reason. Thus in proofs of mathematical theorems, such as in the example under discussion, logic is assumed as an unproblematic foundation for the justification of knowledge.

In summary, the elementary mathematical truth $1 + 1 = 2$ depends for its justification on a mathematical proof.[9] This, in turn, depends on assuming a number of basic mathematical statements (axioms), as well as on the underlying logic. In general, mathematical knowledge consists of statements justified by proofs, which depend on mathematical axioms (and an underlying logic).

This account of mathematical knowledge is essentially that which has been accepted for at least 2,300 years. Early presentations of mathematical knowledge, such as Euclid's *Elements*, are susceptible to the above description and differ from it only by degree. In Euclid, as above, mathematical knowledge is established by the logical deduction of theorems from axioms and postulates (which I include among the axioms). The underlying logic is left unspecified (other than the statement of some axioms concerning the equality relation). The axioms are not regarded as temporarily adopted assumptions, held only for the construction of the theory under consideration. The axioms are considered to be basic truths which need no justification, beyond their own self evidence (Blanché 1966).[10]

Because of this, the account claims to provide absolute grounds for mathematical knowledge. For if the axioms are truths and logical proof preserves truth, then any theorems derived from them must also be truths. This reasoning is implicit, not explicit, in Euclid. However, this claim is no longer accepted because Euclid's axioms and postulates are not considered basic truths which cannot be denied without contradiction. As is well known, the denial of some axioms, most notably the parallel postulate, merely leads to other bodies of geometric knowledge, namely non-Euclidean geometry. As well as the axioms, the proofs of Euclid's *Elements* are now also regarded as flawed and falling short of modern standards of rigor. For they smuggle in notions such as continuity, which is assumed for the accompanying diagrams, even though these have no formal justificatory role in the proofs.

Beyond Euclid, modern mathematical knowledge includes many branches which depend on the assumption of sets of axioms which cannot be claimed to be basic universal truths, for example, the axioms of group theory or of set theory. Maddy (1984) illustrates how modern set theorists add new axioms to Zermelo-Fraenkel set theory and then explore their consequences on a pragmatic basis, rather than regarding the additional axioms as intrinsically true. Henle (1991) also makes this point. However, my claim is that it is not just recondite axioms such as those of set theory that have no claim to be basic and unchallengeable universal truths, but that no such principles exist at all. Even the law of the excluded middle, regarded by philosophers since the time of Aristotle as one of the most basic of all logical principles (Kneale and Kneale 1962), is challenged by a significant group of modern mathematicians and philosophers (the intuitionists), indicating its dubitability and casting doubt on its self-evidence and incontrovertibility.

In what follows I shall be casting further doubt on the infallibility of mathematical knowledge and its foundation in mathematical proof.

THE PHILOSOPHY OF MATHEMATICS

According to Kitcher and Aspray (1988), Frege set the agenda and tone for the modern (i.e., twentieth-century) philosophy of mathematics. Frege ([1884] 1968) adopted the view that the central problem for the philosophy of mathematics is that of identifying the foundations of mathematical knowledge. Basing his analysis on Kant's distinction, Frege argued that mathematical knowledge consists of truths known *a priori*, and that reason alone, in the form of logical proof, provides certain and absolute foundations for it.

Consequently, until recently, twentieth century philosophy of mathematics has been dominated by the quest for absolute foundations for mathematical truth. Of course this can also be viewed as merely the latest expression of an epistemological quest since Plato made an attempt, renewed by Descartes, to find absolute foundations for knowledge in general and for its central pillar, mathematical knowledge.

The aim of this chapter is to offer a critique of this conception and its underlying assumptions. In particular, my main purpose is to expound and criticize the dominant view, for which I shall adopt the term *absolutist*, that mathematical truth is absolutely valid and thus infallible, and that mathematics (with logic) is the one and perhaps the only realm of incorrigible, indubitable, and objective knowledge. I will contrast this with the opposing view, for which I shall adopt the term *fallibilist*, that mathematical truth is fallible and corrigible and should never be regarded as being above revision and correction.

Fallibilism and Absolutism

The first philosopher of mathematics to explicitly state the importance of the absolutist-fallibilist dichotomy is Imre Lakatos (1978b), who relates it to the ancient controversy between dogmatists and skeptics. Lakatos introduced the term *fallibilism*, adapted from Popper's "critical fallibilism," into the philosophy of mathematics.

Lakatos is anticipated by C. S. Peirce's "principle of fallibilism" to the effect that we can know "only in an uncertain and inexact way" (Peirce 1931–58, 5:587) and "there are three things to which we can never hope to attain by reasoning, namely, absolute certainty, absolute exactitude, absolute universality" (1:141).

In philosophy there is some controversy as to what fallibilism means. Haack claims that "fallibilism is a thesis about [1] *our liability to error*, and not a thesis about [2] *the modal status* (possible falsity) *of what we believe*" (1979–80, 309, original emphasis). In contrast O'Hear (1992) suggests that fallibilism is the idea that any human opinions or judgements might turn out false, that is, thesis 2. Following Lakatos I take the view that fallibilism means—as the second of the two views expressed above—that it is theoretically possible that any accepted knowledge including mathematical knowledge may lose its modal status as true or necessary. Such knowledge may have its justificatory warrant rejected or withdrawn (losing its status as knowledge) and be rejected as unwarranted, invalid, or even false.[11]

Lakatos contrasts the term *fallibilism* both with its actual opposite of *infallibilism* (Lakatos 1961, 1976) and more often with an opposing set of perspectives in the philosophy of mathematics that he terms "Euclidean" (Lakatos 1978b, 1976). Infallibilism is synonymous with absolutism, since both mean that mathematical knowledge is indubitable, incorrigible, and infallible. There has been much discussion of the Absolute in the history of philosophy. It occurs metaphysically in the work of Hegel and the idealists Bosanquet, Bradley, and Royce. William James (1912) explicitly uses the term epistemologically when he contrasts absolutism with empiricism. Although the term has been in currency for some time, to the best of my knowledge Confrey (1981) is the first to apply the term in print to the philosophy of mathematics. Recently Harré and Krausz (1996) contrasted absolutism with relativism. Indeed they offer an analysis of different absolutisms and relativisms; this I discuss further in chapter 8.

The absolutist-fallibilist dichotomy distinguishes what in my view is the most important epistemological difference between competing accounts of the nature of mathematics and mathematical knowledge. Indeed, this distinction has pervasive effects through much broader realms than those of philosophy or mathematics alone (Ernest 1991). The distinction parallels that between apriorism and naturalism in the philosophy of mathematics of Kitcher (1984, 1988). Apriorism "is the doctrine that mathematical knowledge is *a priori*," and it "must be obtained from a source different from perceptual experience" (Kitcher 1984, 3). Naturalism opposes this doctrine, and it argues that there are empirical or quasi-empirical sources of justification of mathematical knowledge and that the role of the philosophy of mathematics is to accommodate this and offer a naturalistic account of mathematics. Evidently there is a very close parallel in the two dichotomies; and although there are definitional differences between them, they result in an identical partitioning of schools in the philosophy of mathematics.

Foundationalism in the Philosophy of Mathematics

The target of my critique is any attempt to establish absolutism by means of epistemological foundationalism. The term *foundationalism* is used to describe a number of different perspectives in which belief or knowledge is divided into two parts, foundation and superstructure, and in which the latter depends on the former for its justification, and not vice versa (Alston 1992). Alston points out that some of these senses concern the structure of an individual knower's system of beliefs. This accords with the fact that standard accounts of epistemology often begin discussions of knowledge by referring to individual acts of knowing (Chisholm 1966; Ryle 1949; Woozley 1949). In those acts that conform to Ryle's sense of "knowing that" what is known or grasped is a proposition, the informational content expressed by a sentence. Thus it is possible to consider the content of knowing in traditional epistemological accounts to be knowledge in the form of propositions or sentences. In general this assumption is unwarranted. However, in the case of mathematics (and science) epistemological discussions usually, but not always, refer to knowledge not knowing, that is, to the subject known instead of the knowing subject. Thus the form of knowledge may be taken as the sentence, or a logically organized structure of sentences, the theory. There is more to be said about individual acts of knowing in mathematics, but I shall defer the discussion to later.

Thus a widely adopted assumption in epistemology is that knowledge in any field is represented by a set of propositions, supported with a set of procedures for verifying them or providing a warrant for their assertion. This assumption is remarked upon by Harding (1986), among others, albeit critically. Viewed in this way, mathematical knowledge consists of a set of propositions warranted by proofs. Mathematical proofs are based on deductive reason, comprising chains of necessary inferences. Since it is warranted by reason alone, without recourse to empirical data, mathematical knowledge is understood to be the most infallible and certain of all knowledge, for it avoids the possibilities of error introduced by perception and other empirical sources of knowledge. Traditionally the philosophy of mathematics has seen its task as providing a foundation for this infallibility; that is, providing a system into which mathematical knowledge can be cast to systematically establish its truth. This depends on an assumption, which is widely adopted, implicitly if not explicitly.

The Foundationalist Assumption of the Philosophy of Mathematics: The primary concern of the philosophy of mathematics is establishing that there is, or can be, a systematic and absolutely secure foundation for mathematical knowledge and truth.

This assumption is the basis of foundationalism, the doctrine that the function of the philosophy of mathematics is to provide ultimate and infallible foundations for mathematical knowledge. Foundationalism is bound up with the absolutist view of mathematical knowledge, for it regards the justification of absolutism to be the central problem of the philosophy of mathematics.

Lakatos defines (and critiques) the position he terms "Euclideanism," which is a form of foundationalism modeled on the structure of Euclid's *Elements*. In that system a set of axioms, postulates, definitions, and rules is used to deduce a collection of theorems. Euclideanism similarly seeks to recast mathematical knowledge into a deductive structure based on a finite number of true axioms (or axiom schemes) analogous to Euclid's theory of geometry. This is very similar to, but less general than, the foundationalist assumption or position that I critique. The interpretation of foundationalism adopted here resembles that in Descartes's method. It entails the reconstruction of mathematical knowledge in terms of an absolute foundation and a superstructure infallibly derived from it. The strategy of my critique in this chapter is twofold, reflecting this structure. First, to attack the justificatory basis of the foundation of mathematical knowledge: I shall argue that no absolute foundation for mathematical knowledge can exist. Second, to attack the infallibility of the derivation of the superstructure from it: I shall argue that any such derivation is both fallible and incomplete. This second argument also addresses the position obtained by withdrawing the epistemological assumptions concerning the truth of the foundation. This derived position is a form of hypothetico-deductivism in which the axioms of mathematics are regarded as tentative as opposed to true assumptions. However this position still claims that the derivations of mathematical knowledge are infallible. As I shall show, this revised form remains a version of foundationalism, but one that is based on a different conception of truth.

Kitcher and Aspray (1988) attribute the epistemological and foundational tendency in the philosophy of mathematics to Frege, whom they regard as the founding father and "onlie begetter" of the modern philosophy of mathematics. Frege ([1884] 1968) undertook a thorough critical review of the range of philosophical positions possible, at least to his way of thinking, for arithmetic. This is still regarded as a classic expression of analytic reasoning, perhaps the first such application in the philosophy of mathematics. During the last quarter of the nineteenth century Frege took his main task to be the setting of arithmetic and arithmetical knowledge on a firm foundation. This, as Kitcher and Aspray point out, was a natural extension of the earlier enterprise of constructing firm foundations for analysis pursued by Dedekind, Weierstrass, Heine, and others.[12] Thus Frege installed the foundationalist program, which is essentially epistemological, at the heart of the philosophy of mathematics. He also severely weakened, at least temporarily, the claims of any other programs or approaches to the philosophy of mathematics.

According to Kitcher (1979) and Kitcher and Aspray (1988), Frege ([1884] 1968) analyzed the possible sources of support for the foundations of mathematics into three or four cases. He distinguished between justificatory procedures for mathematics that were *a priori* and *a posteriori*. He criticized and dismissed the possibility that the warrant for mathematical knowledge could be empirical or *a posteriori*.

Given that only two alternatives were admissible to Frege, this meant that the justificatory procedures for mathematics must be *a priori*. He reasoned that only two or three possibilities for justifying *a priori* knowledge are possible. These are, focusing on arithmetic only, as follows. First, that arithmetic is derivable from logic plus the definitions of a special arithmetical vocabulary. Second, that arithmetic is founded on some special *a priori* intuition. Third, a possibility he did not enumerate but treated incidentally, is that arithmetic is not a science with some definite content, but can be represented as a meaningless formal system. Thus, in the alternatives he considered, Frege distinguishes the well springs of empiricism, logicism, intuitionism, and formalism. These possibilities have dominated thinking in the philosophy of mathematics and continue to remain the main possibilities for justifying mathematical knowledge.

ABSOLUTIST VIEWS OF MATHEMATICAL KNOWLEDGE

The absolutist view of mathematical knowledge is that it consists of infallible and absolute truths and represents the unique realm of infallible knowledge (in addition to logic and analytic statements true by virtue of the meanings of terms, such as "All bachelors are unmarried"), which is necessarily true in all possible circumstances and contexts.

Many philosophers, both modern and traditional, hold absolutist views of mathematical knowledge. Thus according to Hempel, in his paper "on the nature of mathematical truth":

> the validity of mathematics derives from the stipulations which determine the meaning of the mathematical concepts, and that the propositions of mathematics are therefore essentially "true by definition." (Feigl and Sellars 1949, 225)

Another proponent of the infallibility of mathematical knowledge is Ayer, who is representative of logical positivism and logical empiricism when he claims that

> truths of mathematics and logic appear to everyone to be necessary and certain. (Ayer 1946, 72)
>
> The certainty of *a priori* propositions depends on the fact that they are tautologies. (Ayer 1946, 16)

The claim that mathematics (and logic) provide necessary knowledge—that is, truth—is based on the deductive method. Logical proof provides the warrant for the assertion of mathematical knowledge, as follows. First of all, the basic statements used in proofs are taken to be true. Mathematical axioms are assumed to be true, for the purposes of developing the system under consideration; mathematical definitions are true by fiat; and logical axioms are accepted as true. Second, the logical rules of inference preserve truth; that is, they allow nothing but truths to be deduced from truths. On the basis of these two facts, every statement in a deductive proof, including its conclusion, is true. Thus, since mathematical theorems are all established by means of deductive proofs, they are all necessary truths. This constitutes the basis of the claim of many philosophers that mathematical truths are infallible truths.

This absolutist view of mathematical knowledge is based on two types of assumptions: those of mathematics, concerning the assumption of axioms and definitions, and those of logic concerning the assumption of axioms, rules of inference, and the formal language and its syntax. These are local or micro-level assumptions. There is also the possibility of global or macro-level assumptions, such as whether logical deduction suffices to establish all mathematical truths, or whether it is always a safe method. I shall subsequently argue that each of these assumptions weakens the claim of infallibility for mathematical knowledge.

The absolutist view of mathematical knowledge encountered problems at the beginning of the twentieth century when a number of antinomies and contradictions were derived in mathematics (Kline 1980; Kneebone 1963; Wilder 1965).[13] In a series of publications Gottlob Frege ([1879] 1967, [1893] 1964) established by far the most rigorous formulation of mathematical logic known to that time, intended as a foundation for mathematical knowledge. Russell ([1902] 1967), however, was able to show that Frege's system was inconsistent. The problem lay in Frege's fifth basic law, which allows a set to be created from the extension of any concept, and for concepts or properties to be applied to this set (Furth 1964). Russell produced his well-known paradox by defining the property of "not being an element of itself." Frege's law allows the extension of this property to be regarded as a set. But then this set is an element of itself if, and only if, it is not; a contradiction. Frege's law could not be dropped without seriously weakening his system, and yet it could not be retained, on pain of contradiction.

Other paradoxes, antinomies, and contradictions emerged in the theories of sets and functions. Such findings have grave implications for the absolutist view of mathematical knowledge. For if mathematics is certain, and all its theorems are certain, how can contradictions (i.e., logical falsehoods) be among its theorems? Since there was no mistake about the appearance of these contradictions, something must be wrong in the foundations of mathematics. The outcome of this crisis was the development of a number of schools in the philosophy of mathematics whose aims were to account for the nature of mathematical knowledge and to reestablish its certainty. The three major schools are known as logicism, formalism, and constructivism (incorporating intuitionism). The tenets of these schools of thought were not fully developed until the twentieth century, but Körner (1960) shows that their philosophical roots can be traced back at least as far as Leibniz and Kant.[14]

Logicism

Logicism is the school of thought that regards pure mathematics as a part of logic. The major proponents of this view, following G. Leibniz's anticipation of it, are G. Frege ([1893] 1964), B. Russell (1919), A. N. Whitehead, and R. Carnap ([1931] 1964). The claims of logicism were most clearly and explicitly formulated by Russell. There are two claims.

1. All the concepts of mathematics can ultimately be reduced to logical concepts, provided that these are taken to include the concepts of set theory or some system of similar power, such as Russell's theory of types.

2. All mathematical truths can be proved from the axioms and rules of inference of logic alone.

The purpose of these claims is clear. If all of mathematics can be expressed in purely logical terms and proved from logical principles alone, then the necessity of mathematical knowledge can be reduced to that of logic. Logic was considered to provide a sure foundation for truth, apart from mistaken attempts, such as Frege's fifth law, that overextended logic. Thus if carried through, the logicist program would provide infallible logical foundations for mathematical knowledge, establishing the absolute validity of mathematics. As a youthful Russell expressed it, "I hoped sooner or later to arrive at a perfected mathematics which should leave no room for doubts" (Russell 1959, 28).

Extending the earlier work of Frege, Peano, and others, Whitehead and Russell (1910–13) were able to establish the first of the two claims by means of elaborate chains of definitions.[15] However logicism foundered on the second claim. Mathematics requires nonlogical rules of inference and axioms such as the principle of mathematical induction (Steiner 1975) and the axioms of infinity and choice.

> But although all logical (or mathematical) propositions can be expressed wholly in terms of logical constants together with variables, it is not the case that, conversely, all propositions that can be expressed in this way are logical. We have found so far a necessary but not a sufficient criterion of mathematical propositions. We have sufficiently defined the character of the primitive *ideas* in terms of which all the ideas of mathematics can be *defined*, but not of the primitive *propositions* from which all the propositions of mathematics can be *deduced*. This is a more difficult matter, as to which it is not yet known what the full answer is.
>
> We may take the axiom of infinity as an example of a proposition which, though it can be enunciated in logical terms, cannot be asserted by logic to be true. (Russell 1919, 202–3)

Russell's claim has been confirmed by subsequent developments. A number of important mathematical axioms are independent, and either the axiom or its negation can be adopted without inconsistency (Cohen 1966). This means that the axioms of mathematics are not eliminable in favor of those of logic. Mathematics is a science with a definite content, and mathematical theorems depend on an irreducible set of mathematical assumptions. Thus the second claim of logicism is refuted.

To overcome this problem Russell retreated to a weaker version of logicism, which has been called "if-thenism." This version obviates the need for mathematical axioms or assumptions by proving a mathematical theorem (T) as before, and then incorporating the conjunction of assumptions used in the proof (A) into an implication statement A→T as its antecedent (Carnap [1931] 1964). This artifice represents the view that mathematics is a hypothetico-deductive system, in which the consequences of assumed axiom sets are explored, without asserting their necessary truth.[16] So if-thenism represents a retreat from the absolutist position of logicism on mathematical knowledge. It is closer to conventionalism, which allows the assumptions of mathematics to be admitted as conventions, truths solely by fiat.

A problem with if-thenism is that many of the axiom sets used as the basis for modern mathematical theories are infinite, due to the use of axiom schemes and first-order logic. Thus both the induction axiom of Peano arithmetic and the axiom of separation of Zermelo-Fraenkel (ZF) set theory, for example, are such schemes (Bell and Machover 1977). Thus mathematical theorems in if-thenism are represented as logically proven implication statements with different finite subsets of axioms conjoined in their antecedents. This leads to misrepresentations such as equivalent results within an axiomatic theory no longer being equivalent, when their antecedents are of different logical power. This is not a refutation of if-thenism. It could be argued that the device is only for use in principle, not in practice, and so such difficulties are irrelevant.

However, the device employed still leads to failure on other grounds. For not all mathematical truths can be expressed in the above way as implication statements. Machover (1983) gives as an example the mathematical truth proved by Gentzen (1936), namely, "Peano Arithmetic is consistent," which is a proposition that cannot be expressed as an implication statement. Another counterexample is the Paris-Harrington proof of a version of Ramsey's theorem (Barwise 1977). This is true in the intended model of Peano arithmetic but not deducible from the axioms (see below). Hence it cannot be represented in the required if-then form. Thus the claim of if-thenism that it provides purely logical foundations for mathematical knowledge is refuted, just as it is with logicism.

A more general objection that holds irrespective of the validity of the two logicist claims also constitutes the major grounds for the rejection of formalism. This arises from Gödel's ([1931] 1967) first incompleteness theorem, which establishes that deductive proof is insufficient for demonstrating all mathematical truths. Hence the successful reduction of mathematical axioms to those of logic would still not suffice for the derivation of all mathematical truths. I shall return to this important result.

A further objection to the whole foundationalist enterprise of logicism, and not to its implementation, concerns the certainty and reliability of the underlying logical apparatus. This security depends on unexamined and, as will be argued, unjustified assumptions. Even if both of the logicist aims were met, the overall program could provide totally reliable foundations for mathematics only if logic were absolutely secure itself. This is assumed but not demonstrated by logicism.

The most important conclusion to be reached is that the logicist program of reducing the necessity of mathematical knowledge to that of logic failed in principle. Logic cannot provide a certain foundation for mathematical knowledge. Irrespective of the security of logic itself, the reliability of mathematical knowledge cannot be reduced to that of pure logic.

Of course, as a mathematical research program in the foundations of mathematics, logicism led to interesting and powerful theories and results. To fulfill the first of its claims alone was a major achievement for modern mathematics. Thus from a mathematical perspective, logicism was very fruitful and led to important successes. The logical definition of many mathematical concepts, Russell's theory of types, the explication of propositional and first-order logic, the early development of proof theory, all of these are among the successes of logicism. But as a philosophy of mathematics, and in particular, as an epistemology seeking to provide mathematical knowledge with an absolute foundation, logicism is without question a failure.

Formalism

In popular terms, formalism is the view that mathematics is a meaningless formal game played with marks on paper, following rules. Traces of a formalist view of mathematics can be found in the writings of Bishop Berkeley ([1710] 1962), but the major proponents of formalism are David Hilbert (1964), early J. von Neumann ([1931] 1964), and H. Curry (1951). Hilbert's formalist program aimed to translate mathematics into uninterpreted formal systems. These were to be shown to be adequate for representing all of mathematics by the creation of a restricted but meaningful ("finitary") metamathematics. Metamathematical proofs would show that the formal counterparts of all mathematical truths could be derived in formal mathematical systems.[17] They would also show, by means of consistency proofs, that the formal systems were safe for representing mathematics.

> The goal of my theory is to establish once and for all the certitude of mathematical methods. (Hilbert 1964, 135)

The formalist thesis comprises two claims:

1. Pure mathematics can be expressed as uninterpreted formal systems in which the truths of mathematics are represented by formal theorems.

2. The safety of these formal systems can be demonstrated in terms of their freedom from inconsistency, by means of metamathematics.

These are very precise claims. However, Kurt Gödel's incompleteness theorems (Gödel [1931] 1967) showed that the program could not be fulfilled. His first theorem showed that not even all the truths of arithmetic can be derived from Peano's axioms (or any larger recursive axiom set). This proof-theoretic result has since been exemplified in mathematics by Paris and Harrington, whose version of Ramsey's theorem is true but not provable in Peano arithmetic (Barwise 1977). Thus the first claim of formalism is refuted in a profound way. It is not possible to translate nontrivial axiomatic theories into formal systems, so that the truths of mathematics are represented by formal theorems.

The second incompleteness theorem showed that in the desired cases consistency proofs require a metamathematics more powerful than the system to be safeguarded; thus there is no safeguard at all. For example, to prove the consistency of Peano arithmetic requires all the axioms of that system and further assumptions, such as the principle of transfinite induction over countable ordinals (Gentzen 1936). Thus it is not possible to prove the consistency of most formal systems of mathematics without, in effect, assuming it. In terms of fulfilling its program, formalism must be regarded as a failure.

The formalist program, had it been successful, would have provided support for an absolutist view of mathematical truth. For formal proof, based in consistent formal mathematical systems, would have provided a touchstone for mathematical truth. However, it can be seen that both the claims of formalism have been refuted. Not all the truths of mathematics can be represented as theorems in formal systems, and furthermore, the systems themselves cannot be guaranteed safe.

Since the refutation of Hilbert's formalist program by Gödel's results, formalists, like the logicists before them, have retreated to a number of weaker positions. Curry (1951), for example, relinquishes the second claim of formalism but maintains a version of the first claim to the effect that mathematics is the science of formal systems. But this position is insupportable as a philosophy of mathematics, for formal systems capture only a proper subset of mathematical knowledge, omitting constructive or recursive mathematics.

Although formalism has been refuted in this way, a number of mathematicians still regard themselves as formalists. For example A. Robinson and P. J. Cohen have written in this vein, as has Henle (1991) more recently. However, few if any try to maintain the refuted foundationalist or epistemological claims of formalism. More often it is the anti-realist ontological position of formalism that commands their support.

Quine (1953a) and Putnam (1972) point out the strong parallel between formalism and nominalism, which latter position originates in the medieval thought of the Schoolmen. Nominalism focuses on the symbolic function of language and, like formalism, denies that it denotes real universals or abstract entities. Thus although there is an illuminating analogy here, it is more relevant to ontological matters than those of epistemology.

Formalism, and related research in the foundations mathematics, can be judged from two perspectives, philosophical and mathematical. Philosophically, however interesting and important it might be historically, formalism is a failed attempt to provide mathematical knowledge with absolute foundations. Particularly in epistemological terms, the formalist program has been shown to be impossible.

In mathematical terms, the program has led to the development and clarification of axiomatic systems, especially set theory, proof theory, and metamathematics, and has contributed to the development of recursion theory, Turing machines, the lambda calculus, and other aspects of formal mathematics vital for the theory of computation. These indicate some of the powerful and important mathematical outcomes stemming at least in part from formalism. Thus mathematically, formalism was a very successful research program.

Constructivism

The constructivist strand in the philosophy of mathematics can be traced back at least as far as Kant and Kronecker (Körner 1960). Kant ([1781] 1961, [1783] 1950), developed an elaborate system of philosophy based on a number of universal mind-given categories of thought, including space and time. He regarded the knowledge of geometry and number as arising from the unfolding of our intuition within these two categories. This gives rise to what he termed the synthetic *a priori* truths of Euclidean geometry and number theory. After Kant's death, the advent of non-Euclidean geometries led many of his followers to abandon the notion that Euclidean geometry consists of synthetic *a priori* truths, derived from the pure intuition of space. However the modern intuitionists such as Brouwer still maintain the other plank of Kant's doctrine. That is, the truths of number are synthetic *a priori* and stem from our basic intuition of time. The attraction of this view is that it anchors mathematical knowledge, at least that of arithmetic, in intuition, guaranteeing its personally meaningful nature.

Against this background, the constructivist program is one of reconstructing mathematical knowledge (and reforming mathematical practice) in order to safeguard it from loss of meaning, and from contradiction. To this end, constructivists reject nonconstructive arguments such as Cantor's proof that the real numbers are uncountable, existence proofs by *reductio ad absurdum*, and the logical laws of double negation and the excluded middle.[18] For these results and modes of reasoning take mathematics beyond what can be constructed intuitively.